Signals and Systems-23EC02

UNIT-IV

Correlation and Sampling Theorem

B.Tech., III-Sem., ECE

1. **Auto Correlation Functions (ACF) and Properties**
2. **Energy Spectral Density (ESD)**
3. **Power Spectral Density (ESD)**
4. **Parseval's Theorem**
5. **Cross Correlation Functions (CCF) and Properties**
6. **Relation between Convolution and Correlation**
7. **Detection of periodic signals in the presence of noise by correlation**
8. **Extraction of signal from noise by filtering**
9. **Sampling Theorem**
   - Graphical Analysis of Sampling Theorem-Impulse Sampling
   - Analytical Analysis of Sampling Theorem
   - Sampling Theorem for Band Limited Signal-Band Pass Sampling
   - Reconstruction of Signal from its Samples
   - Statement of Sampling Theorem
   - Effect of under Sampling-Aliasing Effect
   - Natural and Flat Top Sampling
10. **Solved Problems**
11. **Assignment Questions**
12. **Quiz Questions**

1. Auto Correlation Function (ACF) and Properties:

The measure of similarity between a signal x(t) and time delayed form of the same signal, i.e x(t+τ) is called Auto Correlation Function (ACF). It is represented with Rxx(τ) and it can be computed from the formula

$$Rxx(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \ ; \ if\ x(t)\ is\ aperiodic\ signal$$

$$Rxx(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t-\tau)dt \ ; \ if\ x(t)\ is\ periodic\ signal$$

Properties:

1. Auto Correlation Function Rxx(τ) is even function in τ.

   $$Rxx(-\tau) = Rxx(\tau)$$

2. Total Energy under the signal x(t) can be computed from the Auto Correlation Function Rxx(τ) by substituting τ=0.

   $$E = Rxx(0)$$

3. Maximum value of Auto Correlation Function Rxx(τ) occurs at origin.

   $$Rxx(0) \geq |Rxx(\tau)|$$

4. For an Energy signal, Auto Correlation Function Rxx(τ) and Energy Spectral Density ψ_{xx}(w) form Fourier Transformable pair.

   $$Rxx(t) \underset{IFT}{\overset{FT}{\rightleftarrows}} \Psi xx(w)$$

   Where,

   $$\Psi xx(w) = FT[Rxx(\tau)] = \int_{-\infty}^{\infty} Rxx(\tau)e^{-jw\tau}d\tau$$

5. For a Power signal, Auto Correlation Function Rxx(τ) and Power Spectral Density Sxx(w) form Fourier Transformable pairs.

   $$Rxx(t) \underset{IFT}{\overset{FT}{\rightleftarrows}} Sxx(w)$$

   Where,

   $$Sxx(w) = FT[Rxx(\tau)] = \int_{-\infty}^{\infty} Rxx(\tau)e^{-jw\tau}d\tau$$

2. Energy Spectral Density (ESD):

If x(t) is energy signal, and FT[x(t)] = X(w) with Auto Correlation Function Rxx(τ), then the frequency domain of Rxx(τ) or the squared magnitude spectrum, i.e. $|X(w)|^2$ is called Energy Spectral Density (ESD) or Energy Density Spectrum (EDS). It is represented with $\psi_{xx}(w)$ and it can be computed from the formula;

$$\Psi\text{xx(w)} = FT[Rxx(\tau)] = \int_{-\infty}^{\infty} Rxx(\tau)e^{-jw\tau}d\tau = |\,X(w)\,|^2$$

Proof:

We know that for an energy signal, Auto Correlation Function Rxx(τ) and Energy Spectral Density $\psi_{xx}(w)$ form Fourier Transformable pair. i.e.,

$$\begin{aligned}
\Psi\text{xx(w)} &= FT[Rxx(\tau)] \\
&= \int_{-\infty}^{\infty} Rxx(\tau)e^{-jw\tau}d\tau \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)x(t+\tau)dt\right)e^{-jw\tau}d\tau \\
&= \int_{-\infty}^{\infty} x(t)\left(\int_{-\infty}^{\infty} x(\tau+t)\,e^{-jw\tau}d\tau\right)dt \\
&= \int_{-\infty}^{\infty} x(t)(FT[x(\tau+t)])dt\,, apply\ time\ shifting\ property \\
&= \int_{-\infty}^{\infty} x(t)\,e^{jwt}X(w)dt \\
&= X(w)\int_{-\infty}^{\infty} x(t)\,e^{jwt}dt \\
&= X(w)X^*(w) \\
&= |\,X(w)\,|^2 \\
&= \Psi\text{xx(w)}
\end{aligned}$$

$$\boxed{\Psi\text{xx(w)} = FT[Rxx(\tau)] = \int_{-\infty}^{\infty} Rxx(\tau)e^{-jw\tau}d\tau = |\,X(w)\,|^2}$$

3. Power Spectral Density (PSD):

If x(t) is power signal with Auto Correlation Function Rxx(τ), then the frequency domain of Rxx(τ) is called Power Spectral Density (PSD) or Power Density Spectrum (PDS). It is represented with $S_{xx}(w)$ and it can be computed from the formula;

$$\mathrm{Sxx(w)} = FT[Rxx(\tau)] = \int_{-\infty}^{\infty} Rxx(\tau)e^{-jw\tau}d\tau$$

4. Parseval's Theorem:

If x(t) is energy signal and FT[x(t)] = X(w), then the total energy under the signal x(t) can be computed from x(t) as well as X(w) by using the following relation is called Parseval's theorem.

$$E = \int_{-\infty}^{\infty} |\, x(t)\,|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, X(w)\,|^2\, dw$$

Proof:

We know that the total Energy under the signal x(t) can be computed from the formula

$$E = \int_{-\infty}^{\infty} |\, x(t)\,|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)x^*(t)dt\,;\, IFT[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw\right)^* dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)e^{-jwt}dw\right) dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)\left(\int_{-\infty}^{\infty} x(t)e^{-jwt}dt\right)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)FT[x(t)]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)X(w)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, X(w)\,|^2\, dw$$

$$\boxed{E = \int_{-\infty}^{\infty} |\, x(t)\,|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, X(w)\,|^2\, dw}$$

5. Cross Correlation Function (CCF) and Properties:

The measure of similarity between a signal x(t) and time delayed form of another signal y(t), i.e y(t+τ) is called Cross Correlation Function (CCF). It is represented with Rxy(τ) and it can be computed from the formula

$$Rxy(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt\ ;\ if\ x(t)\ and\ y(t)\ are\ aperiodic\ signal$$

$$Rxy(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt\ ;\ if\ x(t)\ and\ y(t)\ are\ periodic\ signal$$

Properties:

1. If x(t) and y(t) are orthogonal signals, then the Cross Correlation Function Rxy(τ) is zero.

$$Rxy(\tau) = 0$$

or

$$Ryx(\tau) = 0$$

2. Relation between Cross Correlation Functions Rxy(τ) and Ryx(τ) is

$$Rxy(-\tau) = Ryx(\tau)$$

or

$$Rxy(\tau) = Ryx(-\tau)$$

3. Maximum value of the Cross Correlation Function Rxy(τ) occurs at the arithmetic mean of Rxx(0) and Ryy(0).

$$\frac{Rxx(0) + Ryy(0)}{2} \geq |Rxy(\tau)|$$

4. Maximum value of the Cross Correlation Function Rxy(τ) occurs at the geometric mean of Rxx(0) and Ryy(0).

$$\sqrt{Rxx(0)Ryy(0)} \geq |Rxy(\tau)|$$

It is Schwartz inequality.

6. Relation between Convolution and Correlation:

- Convolution and Correlation are two similar and different operations, which is used in almost all signal processing applications to analyze signals and systems in both the time and frequency domains.
- Convolution operation includes four different operations, namely Folding, Shifting, Multiplication and Integration in the case of continues time signals/ Summation in the case of discrete time signals.
- Convolution in continuous time domain can be computed from the formula.

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

- The measure of similarity between a signal x(t) and time delayed form of the same signal, or a different signal y(t) is called Correlation, which includes three different operations, namely Shifting, Multiplication and Integration in the case of continues time signals/ Summation in the case of discrete time signals.
- The measure of similarity between a signal x(t) and time delayed form of the same signal, i.e $x(t+\tau)$ is called Auto Correlation Function (ACF). It is represented with $Rxx(\tau)$ and it can be computed from the formula

$$Rxx(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

- The measure of similarity between a signal x(t) and time delayed form of another signal y(t), i.e $y(t+\tau)$ is called Cross Correlation Function (CCF). It is represented with $Rxy(\tau)$ and it can be computed from the formula

$$Rxy(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt$$

- It is clear from the above analysis that the folding is one extra operation which included in the convolution operation.
- If the second signal y(t) is even, i.e., y(-t)=y(t), then the convolution and correlation operations are same.

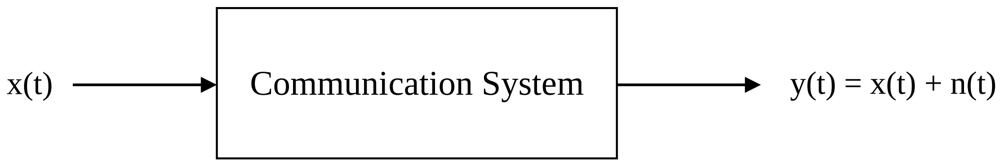
$$x(t) * y(t) = Rxy(\tau), \text{if } y(t) \text{ is even}$$

and

$$x(t) * y(t) \neq Rxy(\tau), \text{if } y(t) \text{ is not even}$$

7. Detection of Periodic Signals in the presence of Noise by Correlation:

- We know that for a distortion less communication system, the output wave shape is exact replica of input wave shape.
- If the noise is added in the communication system, then output may be corrupted or distorted. Now the task is to analyze the system to detect the signal by using correlation functions.
- Let a periodic signal x(t) is transmitted through a communication system and assume the noise n(t) is added and produces an output of y(t).



- Detection of a periodic signal x(t) in the presence of noise n(t) can be analyzed by using auto correlation and cross correlation functions.
- We know that the noise n(t) is random in nature, hence the correlation between a periodic signal x(t) and noise n(t) is zero. i.e. cross correlation between x(t) and n(t) is zero.

$$Rxn(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)n(t-\tau)dt = 0$$

- Now evaluate the auto correlation function of response of a system, y(t) = x(t) + n(t).

$$Ryy(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t)y(t-\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [x(t)+n(t)][x(t-\tau)+n(t-\tau)]dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [x(t)x(t-\tau)+x(t)n(t-\tau)+n(t)x(t-\tau)+n(t)n(t-\tau)]dt$$

$$= Rxx(\tau) + Rxn(\tau) + Rnx(\tau) + Rnn(\tau); where, Rxn(\tau) = Rnx(\tau) = 0$$

$$= Rxx(\tau) + Rnn(\tau)$$

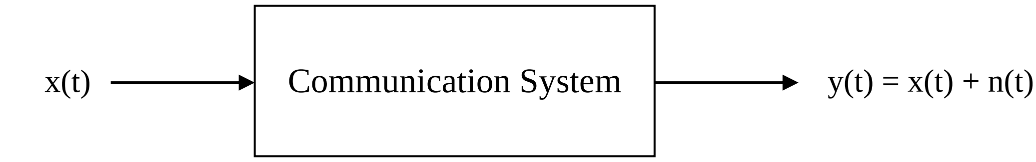
- The noise n(t) is random in nature, and its autocorrelation Rnn(τ) is negligible for higher values of 'τ'. i.e., Rnn(τ) = 0.

$$\boxed{Ryy(\tau) = Rxx(\tau) \text{ or } y(t) = x(t)}$$

- Where Rxx(τ) is periodic because x(t) is periodic and Ryy(τ) is also periodic because the noise n(t) is random in nature, and its autocorrelation Rnn(τ) is negligible. therefore the periodic signal x(t) is said to be detected in the presence of noise.

8. Extraction of Signal from Noise by Filtering:

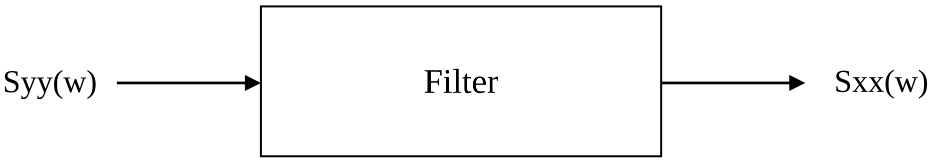
- We know that the correlation functions are used to detect a signal in the presence of noise.



$$whwre, Ryy(\tau) = \; Rxx(\tau) + Rnn(\tau)$$

- Similarly, we can extract the signal from noise by filtering. In general, filters are used to extract required information and attenuate all other unwanted things. In communication, filter can be defined as a frequency selective device for various frequencies of signals.

- Based on frequency response, filters are classified into four types.
  - ✓ Low Pass Filters (LPF)
  - ✓ High Pass Filters (HPF)
  - ✓ Band Pass Filters (BPF)
  - ✓ Band Stop Filters (BSF)
- Depends on type of application use the filter and apply FT[Ryy(τ)] = Syy(w) as input to the filter and take the output, Sxx(w)=FT[Rxx(τ)].



- The ratio between the filter output, Sxx(w) to filter input Syy(w) is called power transfer function of the filter.

$$|H(w)|^2 = \frac{Sxx(w)}{Syy(w)} \Longrightarrow Sxx(w) = |H(w)|^2 Syy(w)$$

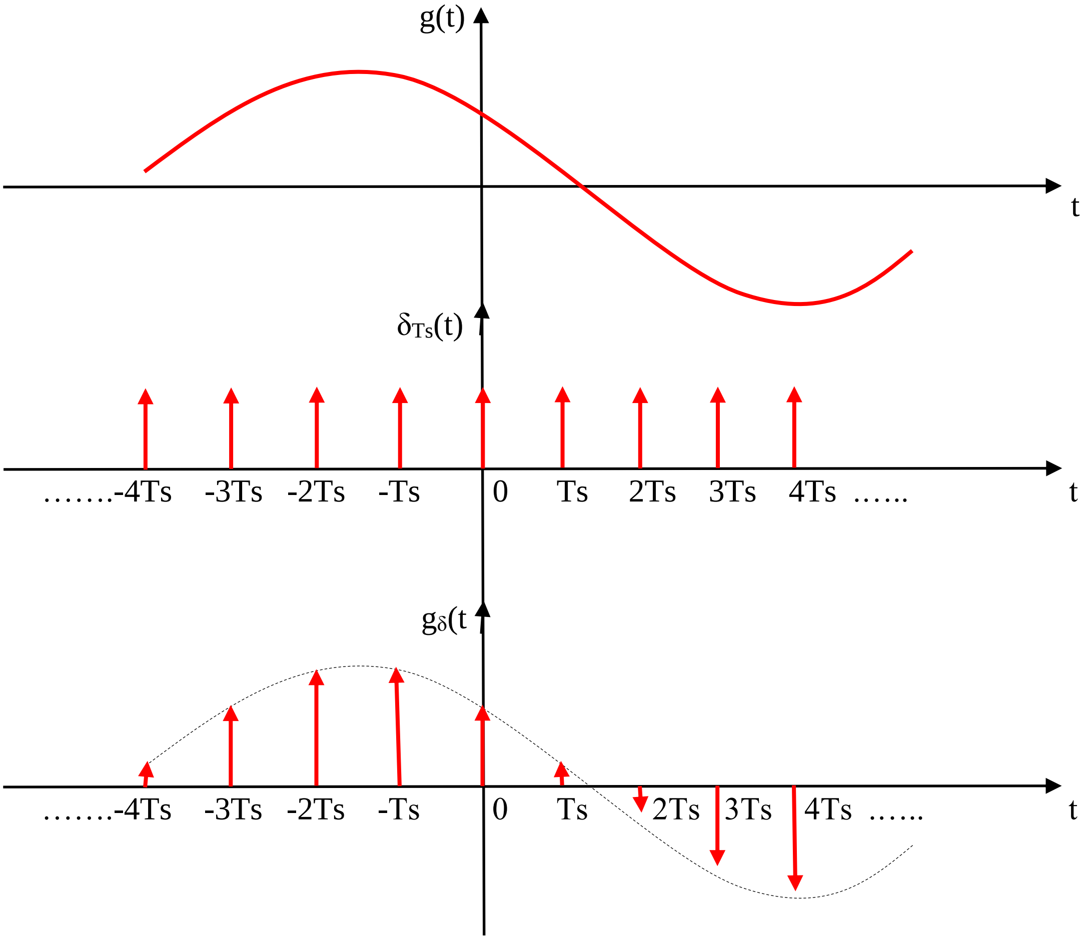
- Computation of Sxx(w) by using above filtering process is called spectral analysis, in this process all unwanted frequency components are attenuated and produces required signal. This is called extraction of signal from noise by filtering.

9. Sampling Theorem:

The process of converting a given continuous time (analog) signal into a discrete time signal is called sampling or sampling process or sampling theorem.

(A)Graphical Analysis of Sampling Theorem-Impulse Sampling:

Discrete time signal can be obtained from the continuous time or analog signal g(t) by multiplying train of impulse signal, $g_\delta(t) = g(t)\delta_{Ts}(t)$. It is called ideal sampling or impulse sampling.



Where,

g(t) : Given continuous time or analog signal

$\delta_{Ts}(t)$: Train of impulse with a time period Ts.

$g_\delta(t)$: Discrete time or sampled form of given g(t)

Ts : Sampling period or Sampling interval

$$g_\delta(t) = g(t)\delta_{Ts}(t), \delta_{Ts}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nTs)$$

$$\Rightarrow g_\delta(t) = g(t) \sum_{n=-\infty}^{\infty} \delta(t - nTs)$$

(B)Analytical Analysis of Sampling Theorem:

From the graphical analysis of sampling theorem,

$$g_\delta(t) = g(t) \sum_{n=-\infty}^{\infty} \delta(t - nTs)$$

Apply Fourier Transform both sides,

$$\Rightarrow FT[g_\delta(t)] = FT\left[g(t) \sum_{n=-\infty}^{\infty} \delta(t - nTs)\right]; use\ FT[x_1(t)x_2(t)] = \frac{X_1(w) * X_2(w)}{2\pi}$$

$$\Rightarrow G_\delta(f)] = FT[g(t)] * FT\left[\sum_{n=-\infty}^{\infty} \delta(t - nTs)\right]$$

$$= G(f) * \frac{1}{Ts} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{Ts}\right)$$

$$= \frac{1}{Ts} \sum_{n=-\infty}^{\infty} G(f) * \delta\left(f - \frac{n}{Ts}\right); use\ x(t) * \delta(t - t_0) = x(t - t_0)$$

$$\Rightarrow G_\delta(f) = \frac{1}{Ts} \sum_{n=-\infty}^{\infty} G\left(f - \frac{n}{Ts}\right)$$

or

$$\Rightarrow G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s)$$

or

$$\Rightarrow G_\delta(w) = w_s \sum_{n=-\infty}^{\infty} G(w - nw_s)$$

$$\Rightarrow G_\delta(w) = w_s \sum_{n=-\infty}^{\infty} G(w - nw_s) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s) = \frac{1}{Ts} \sum_{n=-\infty}^{\infty} G\left(f - \frac{n}{Ts}\right) - - - (2)$$

Where,

fs: Sampling frequency or sampling rate in Hz (fs = 1/Ts)

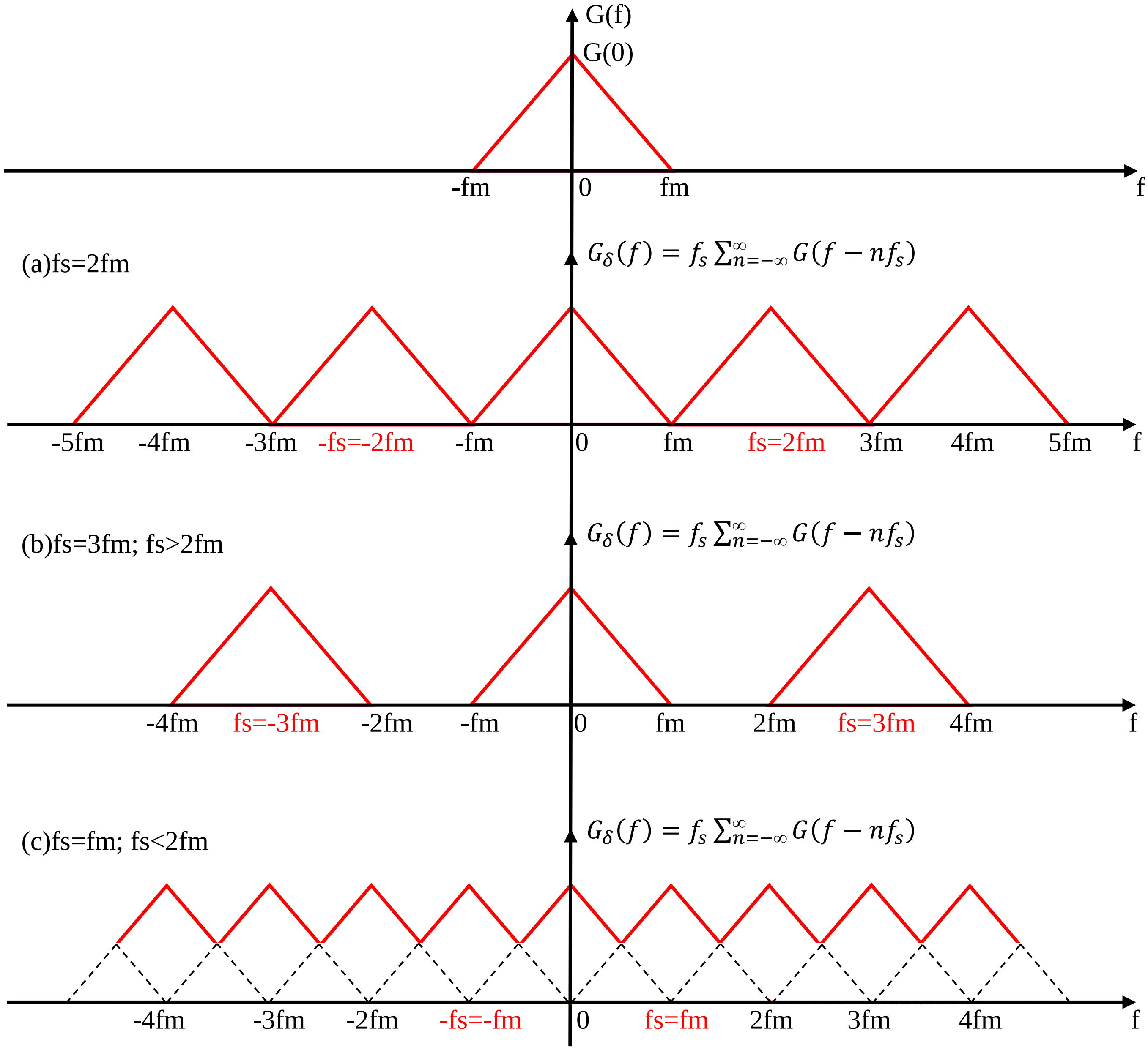
w_s: Sampling frequency in rad/sec

Note: That's the end of graphical and analytical proof for Band Limited Signal under impulse sampling technique.

(C) Sampling Theorem for Band Limited Signal-Band Pass Sampling:

Now the task is to analyze the spectrum $G_\delta(f)$ corresponds to the Band Limited sampled signal $g_\delta(t)$ for the following sampling frequencies by assuming the spectrum of g(t), i.e., G(f) is band limited to 2fm. It is called Band Pass Sampling.

- ✓ fs = 2fm; exact sampling
- ✓ fs = 3fm (fs > 2fm); over sampling
- ✓ fs = fm (fs < 2fm); under sampling



(D) Reconstruction of Signal from its Samples:

It is clear from the above spectral analysis.

- If the sampling frequency, fs =2fm, then all the replicas of G(f) involved in the construction of $G_\delta(f)$ and there is no problem in recovering the original signal g(t) from its sampled version $g_\delta(t)$. In this case, reconstruction of signal from its samples is possible.
- If the sampling frequency, fs =3fm, i.e., fs > 3fm, then all the replicas of G(f) involved in the construction of $G_\delta(f)$ and there is no problem in recovering the original signal g(t) from its sampled version $g_\delta(t)$. In this case also, reconstruction of signal from its samples is possible.
- If the sampling frequency, fs =fm, i.e., fs < 2fm, then the replicas of $G_\delta(f)$ may be overlapped and the original signal g(t) can't be recovered exactly from its sampled version of $g_\delta(t)$. In this case, reconstruction of signal from its samples is not possible.

(E) Statement of Sampling Theorem:

A band limited signal of finite energy can be completely reconstructed from its samples when the samples taken at the rate of $fs \geq 2fm$ samples/sec or it can be completely reconstructed when the sampling interval $Ts \leq 1/2fm$.

Where,

fm : Message Bandwidth

fs : Nyquist Rate (fs = 2fm)

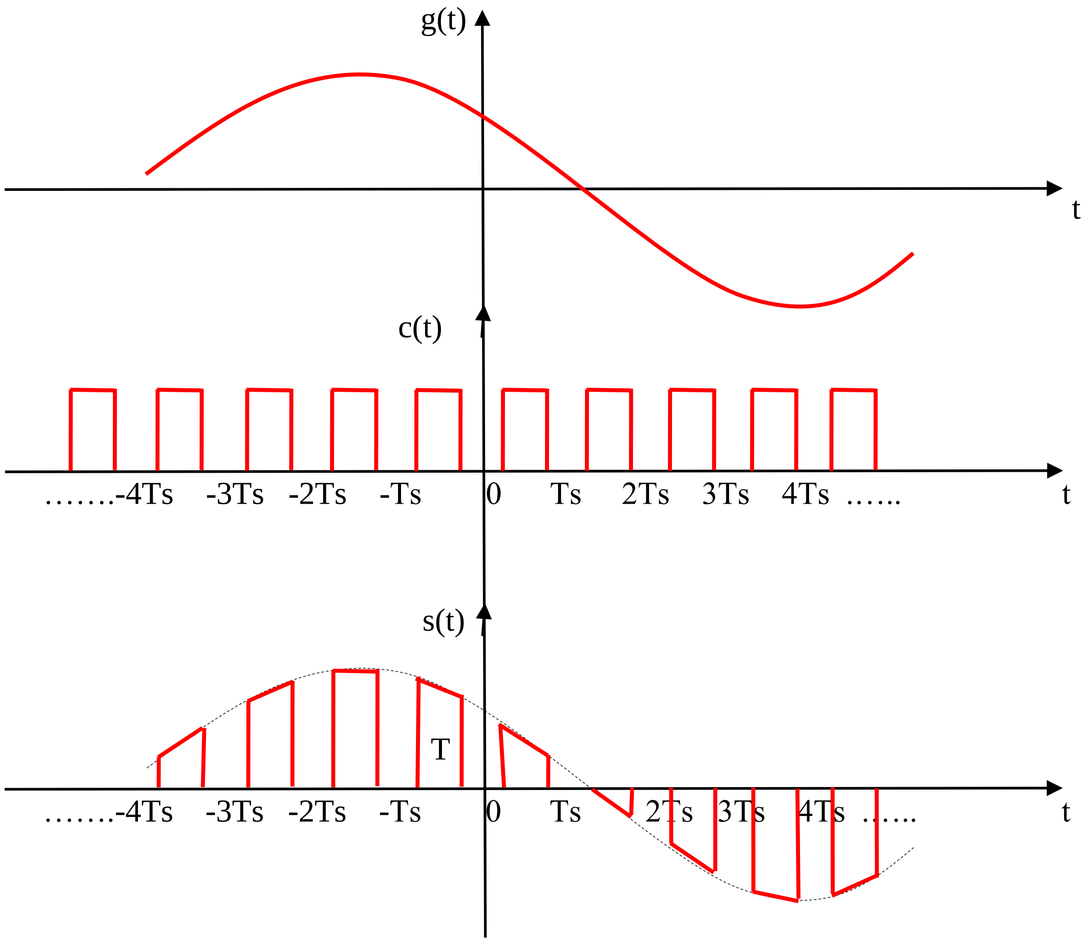
Ts : Nyquist Interval (Ts = 1/fs = 1/2fm)

(F) Effect of under Sampling-Aliasing Effect:

- If the sampling frequency (fs) or sampling rate (1/Ts) is equal to or exceeds the Nyquist rate (2fm), then all the replicas of G(f) involved in the construction of $G_\delta(f)$ and there is no problem in recovering the original signal g(t) from its sampled version $g_\delta(t)$.
- If the sampling frequency (fs) or sampling rate (1/Ts) is less than the Nyquist rate (2fm), then the replicas of $G_\delta(f)$ may be overlapped and the original signal g(t) can't be recovered exactly from its sampled version of $g_\delta(t)$. The loss of information due to this sampling process is called aliasing effect.
- To avoid aliasing, take the sampling frequency $fs \geq 2fm$ or the sampling period $Ts \leq 1/2fm$.

(G) Natural and Flat Top Sampling:

- In the case of ideal or impulse sampling, width of each impulse is negligible. Hence impulse sampling is practically not possible, that's why natural and flat top sampling techniques are used in digital communication.
- In the case of both natural and flat top sampling, the train of pulse is used instead of train of impulse.
- Graphical analysis of natural sampling process as shown, where, s(t)=g(t) c(t).



Where,

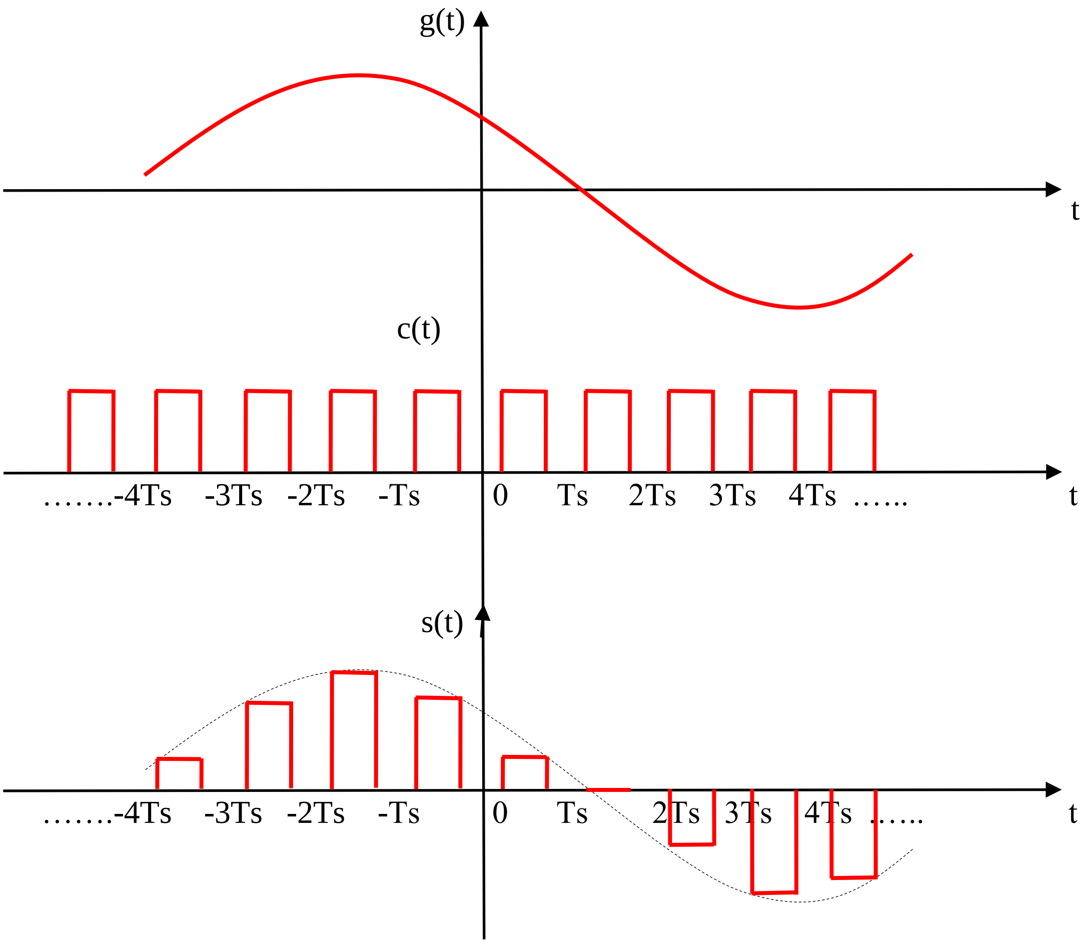
g(t) : Given continuous time or analog signal

c(t) : Train of pulse with a duration T

s(t) : Natural sampled signal

Ts : Sampling period or sampling interval

- Graphical analysis of flat top sampling process as shown, where, s(t)=g(t) c(t).



Where,

g(t) : Given continuous time or analog signal

c(t) : Train of pulse with a duration T

s(t) : Flat top sampled signal

Ts : Sampling period or sampling interval

10. Solved Problem:

(1) Evaluate (a)Auto Correlation Function Rxx(τ) (b)Energy Spectral Density $\psi_{xx}(w)$ of aperiodic signal, $x(t)=ke^{-at}u(t)$.

(a) Auto Correlation Function:

$$Rxx(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

$$= \int_{-\infty}^{\infty} ke^{-at}u(t)ke^{-a(t+\tau)}u(t+\tau)dt$$

$$= k^2\int_{-\infty}^{\infty} e^{-at}e^{-at}e^{-a\tau}u(t)u(t+\tau)dt$$

$$= k^2e^{-a\tau}\int_{-\infty}^{\infty} e^{-2at}u(t)u(t+\tau)dt \ ; \ u(t)=1, t>0 \ \& \ u(t+\tau)=1, t>-\tau$$

$Case-1: If\ \tau<0, then\ u(t)u(t+\tau)=1\ for -\tau<t<\infty$

$$Rxx(\tau) = k^2e^{-a\tau}\int_{-\tau}^{\infty} e^{-2at}dt$$

$$= k^2e^{-a\tau}\frac{e^{-2at}}{-2a}\Big|_{-\tau}^{\infty}$$

$$= k^2e^{-a\tau}\frac{e^{-\infty}-e^{2a\tau}}{-2a}$$

$$= k^2e^{-a\tau}\frac{0-e^{2a\tau}}{-2a}$$

$$= \frac{k^2}{2a}e^{a\tau}, \tau<0$$

$Case-1: If\ \tau>0, then\ u(t)u(t+\tau)=1\ for\ 0<t<\infty$

$$Rxx(\tau) = k^2e^{-a\tau}\int_{0}^{\infty} e^{-2at}dt$$

$$= k^2e^{-a\tau}\frac{e^{-2at}}{-2a}\Big|_{0}^{\infty}$$

$$= k^2e^{-a\tau}\frac{e^{-\infty}-e^{0}}{-2a}$$

$$= k^2e^{-a\tau}\frac{0-1}{-2a}$$

$$= \frac{k^2}{2a}e^{-a\tau}, \tau>0$$

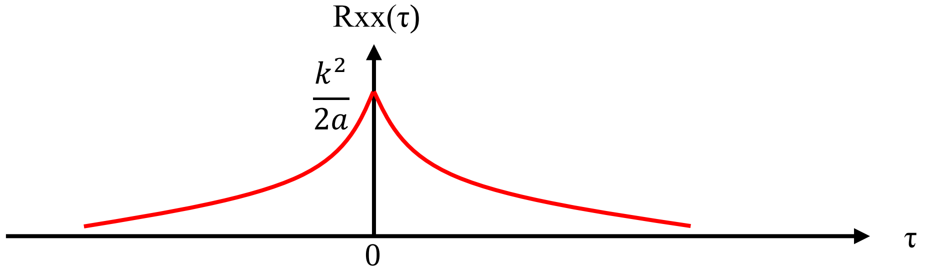
$$Rxx(\tau) = \begin{cases} \frac{k^2}{2a} e^{a\tau}, \tau < 0 \\ \frac{k^2}{2a} e^{-a\tau}, \tau > 0 \end{cases}$$

or

$$Rxx(\tau) = \frac{k^2}{2a} e^{a\tau} u(-\tau) + \frac{k^2}{2a} e^{-a\tau} u(\tau)$$

or

$$Rxx(\tau) = \frac{k^2}{2a} e^{-a|\tau|}$$



(b) Energy Spectral Density:

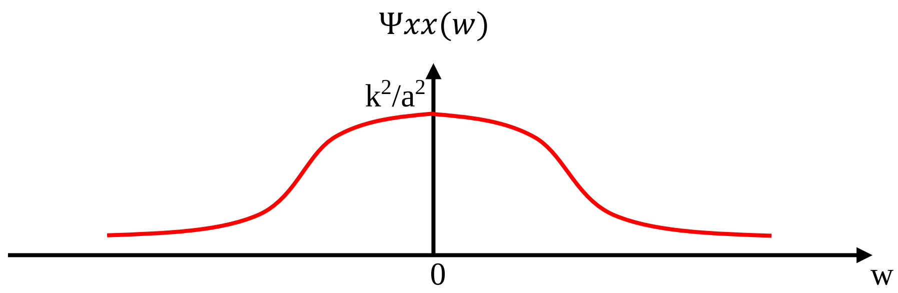
$$\Psi xx(w) = FT[Rxx(\tau)]$$

$$= FT\left[\frac{k^2}{2a} e^{-a|\tau|}\right]$$

$$= \frac{k^2}{2a} FT\left[e^{-a|\tau|}\right]$$

$$= \frac{k^2}{2a} \frac{2a}{a^2 + w^2}$$

$$= \frac{k^2}{a^2 + w^2}$$



(2) Evaluate (a)Auto Correlation Function Rxx(τ) (b)Power Spectral Density Sxx(w) of periodic signal, x(t) = ACos(w_0t+θ).

(a) Auto Correlation Function:

$$Rxx(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} ACos(w_0 t+\theta)ACos(w_0(t+\tau)+\theta)dt$$

$$= \frac{A^2}{2}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 2Cos(w_0 t+\theta)Cos(w_0 t+w_0\tau+\theta)dt$$

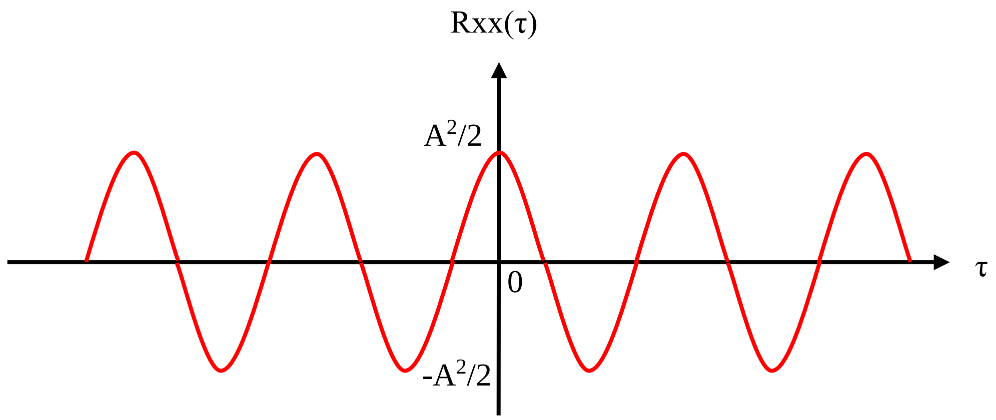
$$= \frac{A^2}{2}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (Cos(2w_0 t+w_0\tau+2\theta)+Cos(w_0\tau))dt$$

$$= \frac{A^2}{4}\lim_{T\to\infty} \frac{1}{T}\left(\int_{-T}^{T} Cos(2w_0 t+w_0\tau+2\theta)dt+\int_{-T}^{T} Cos(w_0\tau)\,dt\right)$$

$$= \frac{A^2}{4}\lim_{T\to\infty} \frac{1}{T}\left(0+2TCos(w_0\tau)\right)$$

$$= \frac{A^2}{2}\lim_{T\to\infty}\left(Cos(w_0\tau)\right)$$

$$= \frac{A^2}{2}Cos(w_0\tau)$$



(b) Power Spectral Density:

$$Sxx(w) = FT[Rxx(\tau)]$$

$$= FT\left[\frac{A^2}{2}Cos(w_0\tau)\right]$$

$$= \frac{A^2}{2}FT[Cos(w_0\tau)]$$

$$= \frac{\pi A^2}{2}\left(\delta(w+w_0)+\delta(w-w_0)\right)$$

(3) Evaluate (a)Auto Correlation Function Rxx(τ) (b)Power Spectral Density Sxx(w) of periodic signal, x(t) = ASin(w_0t+θ).

(a) Auto Correlation Function:

$$Rxx(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} ASin(w_0 t+\theta)ASin(w_0(t+\tau)+\theta)dt$$

$$= \frac{A^2}{2}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 2Sin(w_0 t+\theta)Sin(w_0 t+w_0\tau+\theta)dt$$

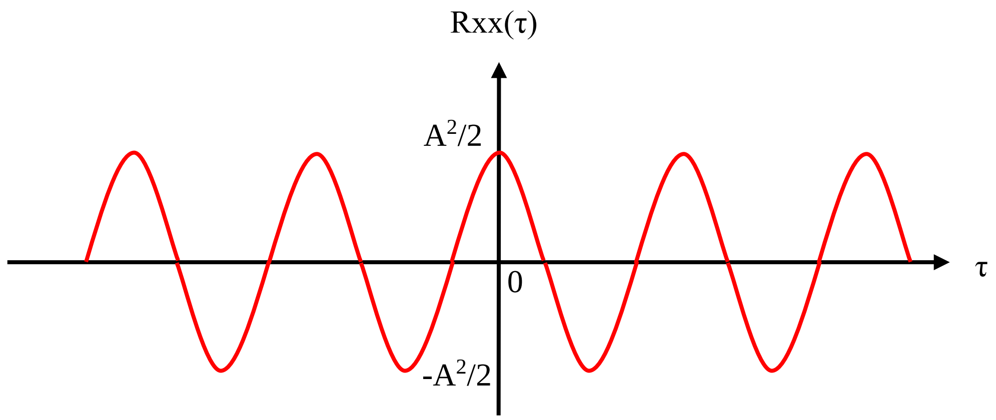
$$= \frac{A^2}{2}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (Cos(w_0\tau) - Cos(2w_0 t+w_0\tau+2\theta)+)dt$$

$$= \frac{A^2}{4}\lim_{T\to\infty} \frac{1}{T}\left(\int_{-T}^{T} Cos(w_0\tau)\,dt - \int_{-T}^{T} Cos(2w_0 t+w_0\tau+2\theta)dt\right)$$

$$= \frac{A^2}{4}\lim_{T\to\infty} \frac{1}{T}(2TCos(w_0\tau) - 0)$$

$$= \frac{A^2}{2}\lim_{T\to\infty}\left(Cos(w_0\tau)\right)$$

$$= \frac{A^2}{2}Cos(w_0\tau)$$



(b) Power Spectral Density:

$$Sxx(w) = FT[Rxx(\tau)]$$

$$= FT\left[\frac{A^2}{2}Cos(w_0\tau)\right]$$

$$= \frac{A^2}{2}FT[Cos(w_0\tau)]$$

$$= \frac{\pi A^2}{2}\left(\delta(w+w_0) + \delta(w-w_0)\right)$$

(4) Verify the Parseval's theorem for a signal $x(t) = 3e^{-2t}u(t)$

$$We\ know\ that, FT[x(t)] = X(w) = FT[3e^{-2t}u(t)] = \frac{3}{2+jw}$$

$$Parsevall's\ Theorem, E = \int_{-\infty}^{\infty} |\, x(t)\,|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, X(w)\,|^2\, dw$$

$$LHS = \int_{-\infty}^{\infty} |\, 3e^{-2t}u(t)|^2 dt$$

$$= \int_{0}^{\infty} 9e^{-4t} dt$$

$$= 9\frac{e^{-4t}}{-4}\Big|_{0}^{\infty}$$

$$= \frac{9}{4}$$

$$RHS = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{3}{2+jw}\right|^2 dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{9}{4+w^2} dw$$

$$= \frac{9}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2^2+w^2} dw$$

$$= \frac{9}{2\pi}\frac{1}{2}\tan^{-1}\frac{w}{2}\Big|_{-\infty}^{\infty}$$

$$= \frac{9}{4\pi}(\tan^{-1}\infty - \tan^{-1}(-\infty))$$

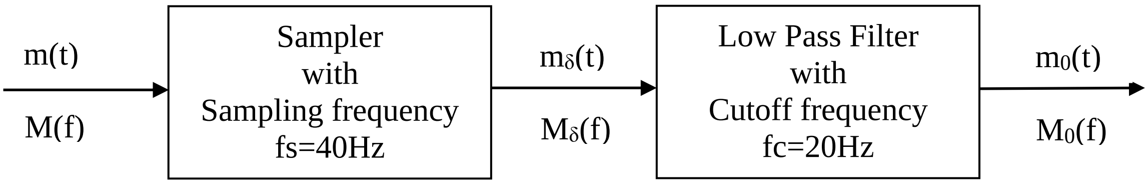
$$= \frac{9}{4\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= \frac{9}{4\pi}\pi$$

$$= \frac{9}{4}$$

LHS = RHS

(5) Evaluate the output of a low pass filter, when the input message signal m(t) = Cos(2πfmt) with fm = 50Hz is sampled at sampling frequency fs = 40Hz is transmitted through low pass filter with cutoff frequency fc = 20Hz.



Input message signal

$m(t) = Cos(2\pi f_m t);\ FT[Cos(w_0 t)] = \pi(\delta(w + w_0) + \delta(w - w_0))$

$\Rightarrow M(f) = \delta(f + fm) + \delta(f - fm)$

$= \delta(f + 50) + \delta(f - 50)$

Sampled form of M(f)

$$M_\delta(f) = fs \sum_{n=-\infty}^{\infty} M(f - nfs)$$

$$= 40 \sum_{n=-\infty}^{\infty} M(f - 40n)$$

$$= \cdots + 40M(f + 40) + 40M(f) + 40M(f - 40) + \cdots$$

$$= \cdots + 40\delta(f + 90) + 40\delta(f - 10) + 40\delta(f + 50) + 40\delta(f - 50)$$
$$+40\delta(f + 10) + 40\delta(f - 90) + \cdots$$

$$= \cdots + 40[\delta(f + 90) + \delta(f - 90)] + 40[\delta(f + 50) + 40\delta(f - 50)]$$
$$+40[\delta(f + 10) + \delta(f - 10)] + \cdots$$

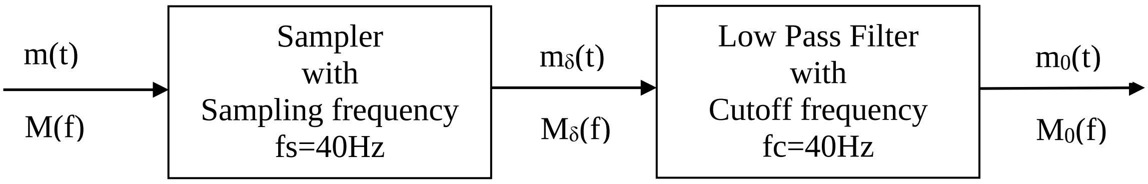
Output of a LPF with cutoff frequency f_c=20Hz is

$M_0(f) = 40[\delta(f + 10) + \delta(f - 10)]$

$\Rightarrow m_0(t) = 40Cos(2\pi \times 10t)$

$= 40Cos(20\pi t)$

(6) Evaluate the output of a low pass filter, when the input message signal m(t) = Cos(2πfmt) with fm = 50Hz and it is sampled at a sampling frequency fs = 40Hz is transmitted through a low pass filter with cutoff frequency fc = 40Hz.



Input message signal

$m(t) = Cos(2\pi f_m t)$

$\Rightarrow M(f) = \delta(f + fm) + \delta(f - fm)$

$= \delta(f + 50) + \delta(f - 50)$

Sampled form of M(f)

$$M_\delta(f) = fs \sum_{n=-\infty}^{\infty} M(f - nfs)$$

$$= 40 \sum_{n=-\infty}^{\infty} M(f - 40n)$$

$= \ldots + 40M(f + 80) + 40M(f + 40) + 40M(f) + 40M(f - 40) + 40M(f - 80) + \ldots$

$= \ldots + 40\delta(f - 30) + 40\delta(f - 10) + 40\delta(f + 50) + 40\delta(f - 50) + 40\delta(f + 10)$
$+ 40\delta(f + 30)$

$= \cdots + 40[\delta(f + 50) + \delta(f - 50)] + 40[\delta(f + 30) + 40\delta(f - 30)]$
$+ 40[\delta(f + 10) + \delta(f - 10)] + \ldots$

Output of a LPF with cutoff frequency $f_c = 40$Hz is

$M_0(f) = 40[\delta(f + 30) + \delta(f - 30)] + 40[\delta(f + 10) + \delta(f - 10)]$

$\Rightarrow m_0(t) = 40Cos(2\pi \times 30t) + 40Cos(2\pi \times 10t)$

$= 40Cos(60\pi t) + 40Cos(20\pi t)$

11. Assignment Questions

1. Evaluate the autocorrelation function $R_{XX}(\tau)$ corresponds to the power spectral density
$$S_{XX}(w) = \frac{157 + 12w^2}{(16 + w^2)(9 + w^2)}$$

2. Evaluate the cross-correlation function, $R_{XY}(\tau)$ between the stochastic process X(t) and Y(t), which has the cross power spectral density
$$S_{XY}(w) = \begin{cases} 1 + jw, |w| \leq 1 \\ 0, otherwise \end{cases}$$

3. A stochastic process X(t) has an autocorrelation function, $R_{XX}(\tau) = 3e^{-2|\tau|}$.
(i)Evaluate the power spectral density, $S_{XX}(w)$.
(ii)Draw the power spectrum.

12. Quiz Questions

(16) The autocorrelation function of a periodic signal x(t) is

(A) $\boldsymbol{R_{xx}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt}$

(B) $\boldsymbol{R_{xx}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)x(t+\tau)dt}$

(C) $\boldsymbol{R_{xx}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(-t+\tau)dt}$

(D) $R_{xx}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)x(-t+\tau)dt$

(19) The measure of similarity between a signal x(t) and delayed form of another signal y(t) is

(A) Convolution

(B) Auto Correlation

(C) Cross Correlation

(D) Energy Spectral Density

(20) Autocorrelation function of aperiodic signal x(t) is

(A) $\boldsymbol{R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt}$

(B) $\boldsymbol{R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(-t+\tau)dt}$

(C) $\boldsymbol{R_{xx}(\tau) = \int_{-\infty}^{\infty} x(\tau)x(t+\tau)dt}$

(D) $R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t-\tau)x(t+\tau)dt$

(23) Cross correlation function between x(t) & y(t) is

(A) $\boldsymbol{R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t-\tau)dt}$

(B) $\boldsymbol{R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(-t+\tau)dt}$

(C) $\boldsymbol{R_{xy}(\tau) = \int_{-\infty}^{\infty} x(\tau)y(t+\tau)dt}$

(D) $R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t-\tau)y(t+\tau)dt$

(24) If x(t) and y(t) are orthogonal, then the cross-correlation function is

(A) 0

(B) 1

(C) -1

(D) Infinity

(25) Choose the correct property of autocorrelation function

(A) $R_{xx}(-\tau) = R_{xx}(\tau)$

(B) $R_{xx}(0) = Energy\ of Signal$

(C) $R_x(0) \geq |R_x(\tau)|$

(D) All the above

(26) The Auto correlation function and Power spectral density form

(A) Laplace Transform Pair

(B) Z-Transform Pair

(C) Fourier Transform Pair

(D) All the above

(27) Which are Fourier Transformable pairs?

(A) ESD & ACF for an Energy signal

(B) PSD & ACF for Power signal

(C) Impulse and Constant signal

(D) All the above

(28) Auto correlation function of $x(t) = 2Cos(2t)u(t)$ is

(A) $R_{xx}(\tau) = 2Sin(2\tau)$

(B) $R_{xx}(\tau) = 2Cos(2\tau)$

(C) $R_{xx}(\tau) = 4Sin(2\tau)$

(D) $R_{xx}(\tau) = 4Cos(2\tau)$

(29) Auto correlation function of $x(t) = 2Sin(2t)u(t)$ is

(A) $R_{xx}(\tau) = 2Sin(2\tau)$

(B) $R_{xx}(\tau) = 2Cos(2\tau)$

(C) $R_{xx}(\tau) = 4Sin(2\tau)$

(D) $R_{xx}(\tau) = 4Cos(2\tau)$

(25) The process of converting a continuous-time signal into a discrete-time signal is called

(A) Sampling

(B) Quantization

(C) Both (A) & (B)

(D) None of the above

(26) The time interval between any two adjacent samples of sampled signal is

(A) Nyquist Rate

(B) Nyquist Interval

(C) Sampling period

(D) Both (B) and (C)

(27) What is the minimum sampling frequency required for the sampling process with no distortion, given analog signal $x(t) = Cos(100\pi t) + 6Sin(180\pi t)$ is

(A) 100 Hz

(B) 90 Hz

(C) 180 Hz

(D) 360 Hz

(30) A band limited signal with highest frequency content of 1000 Hz is undergoing sampling at uniform intervals. For recovery of the signal in an unambiguous way, the sampling frequency should be necessarily greater than

(A) 2000 Hz

(B) 1500 Hz

(C) 1000 Hz

(D) 500

(31) A band-limited signal is sampled at the Nyquist rate. The signal can be recovered by passing the samples through

(A) High Pass Filter

(B) Low Pass Filter

(C) Band Pass Filter

(D) Band Stop Filter

(32) The process of converting an analog signal into a digital signal involves

(A) Sampling

(B) Quantization

(C) Both (A) & (B)

(D) None of the above

(33) Consider the signal m(t) with spectrum M(f) which is sampled with an ideal impulse train corresponding to sampling frequency fs. The spectrum of resulting sampled signal is

$$(A) FT[m_\delta(t)] = w_s \sum_{n=-\infty}^{\infty} \delta(w - nw_s)$$

$$(B) FT[m_\delta(t)] = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

$$(C) FT[m_\delta(t)] = \frac{1}{Ts} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{Ts}\right)$$

$(\boldsymbol{D})$ *All* the above are correct

(49) What is the minimum sampling frequency fs required for no distortion if the maximum message frequency is fm.

fs = 2fm

(50) Consider the signal m(t) with spectrum M(f) which is sampled with an ideal impulse train corresponding to sampling frequency fs. The spectrum of resulting sampled signal is

$$FT[m_\delta(t)] = w_s \sum_{n=-\infty}^{\infty} M(w - nw_s) = f_s \sum_{n=-\infty}^{\infty} M(f - nf_s) = \frac{1}{Ts} \sum_{n=-\infty}^{\infty} M\left(f - \frac{n}{Ts}\right)$$

(51) What is the minimum sampling frequency fs required for no distortion if the message signal m(t)=Cos(2πfmt) with fm=50Hz.

fs = 2fm = 2x50 = 100Hz

(52) What is the minimum sampling frequency fs required for no distortion if the message signal m(t)= Cos(3πfmt) + Cos(2πfmt)+ Cos(πfmt).

m(t)= Cos(2π(3fm/2)t) + Cos(2π(fm)t)+ Cos(2π(fm/2)t).

Minimum sampling frequency fs=2 x Maximum message frequency

fs = 2(3fm/2) = 3fm